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The Determination of the Main Indicators of a Production Function Using the Bernoulli Equations

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Abstract: Production functions are an essential tool for analysis of production processes. The indicators of marginal production, marginal rates of substitution, elasticities of production and the marginal elasticity of technical substitution characterized, from different point of view, the behavior of the production under the action of factors of labor or capital. This paper presents a new way of determining using the first-order differential equation of Bernoulli type, giving also a useful tool for the creation of new production functions.

Keywords: production function; marginal rate of substitution; elasticity

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1Introduction

In the analysis of production processes, fundamentally important are the production functions, which are accompanied by a series of indicators that provide useful information in the economic analysis. The production functions approach can be based on practical needs ([10]) or on the conditions of their indicators ([1], [3], [9], [11], [12], [13]). Another approach can be done in terms of purely geometric properties ([4], [5]), but also through the generalization of existing functions and then putting in obviousness their common characteristics ([6], [8]).

In this paper, we will broach the problem of determining the main indicators from a different point of view. Even if they come from different economical considerations, we will construct a first-order differential equation of Bernoulli type, whose coefficients will allow the immediate determination of those indicators.

At first glance, the question arises: how to use such an approach? The problem has an immediate response, although not very visible. The differential equation, by assigning different expressions to the composing functions, can be a true generator of production functions!

Let therefore a production function $Q:(0,\infty)^2\to\mathbf{R}_+$, $(K,L)\to Q(K,L)$ homogeneous of first degree.

Let note, also $\chi=\frac{K}{L}$ - the technical endowment of labor. We have:

$$Q(K,L)=Q\left(L\frac{K}{L},L\right)=LQ\left(\frac{K}{L},1\right)=L\cdot q(\chi)$$

where $q(\chi)=Q(\chi,1)$.

The partial first-order derivations of Q, are obtained easily:

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$$\frac{\partial Q(K,L)}{\partial K}=L\frac{\partial q(\chi)}{\partial K}=L\frac{\partial q(\chi)}{\partial \chi}\frac{\partial \chi}{\partial K}=Lq'(\chi)\frac{1}{L}=q'(\chi)$$

$$\frac{\partial Q(K,L)}{\partial L}=\frac{\partial (Lq(\chi))}{\partial L}=q(\chi)+L\frac{\partial q(\chi)}{\partial \chi}\frac{\partial \chi}{\partial L}=q(\chi)-Lq'(\chi)\frac{K}{L^2}=q(\chi)-\chi q'(\chi)$$

Since the production function is strictly increasing in each variable, results:

$\frac{\partial Q(K,L)}{\partial K}>0$, $\frac{\partial Q(K,L)}{\partial L}>0$ therefore: $q'(\chi)>0$ and $q(\chi)-\chi q'(\chi)>0$. The function q is strictly increasing and $\frac{q'(\chi)}{q(\chi)}<\frac{1}{\chi}$. Noting $y(\chi)=\frac{q'(\chi)}{q(\chi)}$ follows that: $y(\chi)<\frac{1}{\chi}$.

We have now:

$$\frac{\partial Q^2(K,L)}{\partial K^2}=\frac{\partial q'(\chi)}{\partial K}=\frac{\partial q'(\chi)}{\partial \chi}\frac{\partial \chi}{\partial K}=\frac{1}{L}q''(\chi)$$

$$\frac{\partial Q^2(K,L)}{\partial K\partial L}=\frac{\partial q'(\chi)}{\partial L}=\frac{\partial q'(\chi)}{\partial \chi}\frac{\partial \chi}{\partial L}=-\frac{K}{L^2}q''(\chi)=-\frac{1}{L}\chi q''(\chi)$$

$$\frac{\partial Q^2(K,L)}{\partial L^2}=\frac{\partial\left(q(\chi)-\chi q'(\chi)\right)}{\partial L}=\frac{\partial\left(q(\chi)-\chi q'(\chi)\right)}{\partial \chi}\frac{\partial \chi}{\partial L}=\frac{K}{L^2}\chi q''(\chi)=\frac{1}{L}\chi^2 q''(\chi)$$

The second differential of Q is:

$$d^2Q=\frac{1}{L}q''(\chi)dK^2-\frac{2}{L}\chi q''(\chi)dKdL+\frac{1}{L}\chi^2 q''(\chi)dL^2=\frac{1}{L}q''(\chi)\left(dK-\chi dL\right)^2$$

Since the function Q is concave, we have $d^2Q<0$, therefore $q''(\chi)<0$ that is q is concave.

The conditions to be fulfilled for the existence of the production function are therefore:

$$q(\chi)>0,\ q'(\chi)>0,\ 0<y(\chi)<\frac{1}{\chi},\ q''(\chi)<0$$

Let us note now that: $y'(\chi)=\frac{q''(\chi)q(\chi)-q'^2(\chi)}{q^2(\chi)}=\frac{q''(\chi)}{q(\chi)}-y^2(\chi)$ therefore:

$$\begin{cases} q'(\chi)=q(\chi)y(\chi) \\ q''(\chi)=q(\chi)\left(y'(\chi)+y^2(\chi)\right) \end{cases}$$

Also: $y'(\chi)=\frac{q''(\chi)}{q(\chi)}-y^2(\chi)<0$ or else: $y(\chi)(g(\chi)+h(\chi)y(\chi))<0$ from where: $g(\chi)+h(\chi)y(\chi)<0$.

The main indicators associated with the production function are:

- **the labor productivity**:

$$w_L=\frac{Q(K,L)}{L}=q(\chi)$$

- **the productivity of capital**:

$$w_K=\frac{Q(K,L)}{K}=\frac{q(\chi)}{\chi}$$

- **the marginal production of labor**:

$$\eta_L=\frac{\partial Q(K,L)}{\partial L}=q(\chi)-\chi q'(\chi)=q(\chi)\left(1-\chi\frac{q'(\chi)}{q(\chi)}\right)=q(\chi)(1-\chi y(\chi))$$

- **the marginal production of capital**:

$$\eta_K=\frac{\partial Q(K,L)}{\partial K}=q'(\chi)=y(\chi)q(\chi);$$

- **the marginal rate of substitution between K and L**:

$$RMS(L,K)=\frac{\eta_L}{\eta_K}=\frac{q(\chi)}{q'(\chi)}-\chi=\frac{1}{y(\chi)}-\chi;$$

- **the marginal rate of substitution between L and K**:

$$RMS(K,L)=\frac{\eta_K}{\eta_L}=\frac{q'(\chi)}{q(\chi)-\chi q'(\chi)}=\frac{y(\chi)}{1-\chi y(\chi)};$$

- **the output elasticity with respect to capital**:

$$\varepsilon_K=\frac{\eta_K}{w_K}=\frac{\chi q'(\chi)}{q(\chi)}=\chi y(\chi);$$

- **the output elasticity with respect to the labor**:

$$\varepsilon_L=\frac{\eta_L}{w_L}=1-\frac{\chi q'(\chi)}{q(\chi)}=1-\chi y(\chi);$$

- **the elasticity of marginal rate of technical substitution**:

$$\sigma=\frac{\frac{\partial RMS(K,L)}{\partial \chi}}{\frac{RMS(K,L)}{\chi}}=\frac{\chi q(\chi)q''(\chi)}{q'(\chi)(q(\chi)-\chi q'(\chi))}=\frac{\chi(y'(\chi)+y^2(\chi))}{y(\chi)(1-\chi y(\chi))}.$$

2. General Mathematical Results

Lemma 1

Let a function $y:(0,\infty)\to\mathbf{R}$, of class C^1 and the equation:

$$y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$$

where $g,h:(0,\infty)\to\mathbf{R}$, continous on $(0,\infty)$, $\lim_{\chi\to 0} y(\chi)=\infty$. Then:

$$y(\chi)=-\frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}$$

Proof. With the substitution $y(\chi)=\frac{1}{z(\chi)}$, we have: $y'(\chi)=-\frac{z'(\chi)}{z^2(\chi)}$ from where:

$$z'(\chi)=-g(\chi)z(\chi)-h(\chi)$$

Let the attached homogeneous linear equation: $z'(\chi) = -g(\chi)z(\chi)$. Because $\frac{dz}{z} = -g(\chi)d\chi$ follows: $\ln z = -\int g(\chi)d\chi + C$ therefore: $z = Ce^{-\int g(\chi)d\chi}$. Applying the method of variation constants and considering $z(\chi) = C(\chi)e^{-\int g(\chi)d\chi}$ we obtain, after replacing in the linear equation: $C'(\chi)e^{-\int g(\chi)d\chi} = -h(\chi)$ from where: $C(\chi) = -\int h(\chi)e^{\int g(\chi)d\chi}d\chi + D$ therefore:

$$z(\chi) = e^{-\int g(\chi)d\chi}\left(D - \int h(\chi)e^{\int g(\chi)d\chi}d\chi\right),\ D\in \mathbf{R}$$

Finally:

$$y(\chi)=\frac{e^{\int g(\chi)d\chi}}{D - \int h(\chi)e^{\int g(\chi)d\chi}d\chi},\ D\in \mathbf{R}$$

Writing the above relationship as: $y(\chi)=\frac{e^{\int_0^{\chi} g(t)dt}}{D - \int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}$, $D\in \mathbf{R}$ we have: $\lim_{\chi\to 0} y(\chi) = \frac{1}{D} = \infty$ therefore D=0. We have now the desired formula:

$$y(\chi)=-\frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}$$

Q.E.D.

Lemma 2

Considering a function $q:(0,\infty)\to\mathbf{R}$, q of class C^2 and $y:(0,\infty)\to\mathbf{R}$, $y(\chi)=\frac{q'(\chi)}{q(\chi)}$ such that: $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$ then: $q(\chi) = Ge^{-\int \frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}d\chi}$, $G\in \mathbf{R}_+$.

Proof. After lemma 1, we have: $\frac{q'(\chi)}{q(\chi)}=y(\chi)=-\frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}$ therefore: $(\ln q(\chi))' = -\frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}$

from where: $\ln q(\chi) = -\int \frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt}ds}d\chi + G$ and, finally:

$$q(\chi) = Ge^{-\int \frac{e^{\int_0^{\chi} g(t)dt}}{\int_0^{\chi} h(s)e^{\int_0^{s} g(t)dt} ds} d\chi}, \ G \in \mathbf{R}_+$$

Q.E.D.

3. The Characterization of Economic Indicators with Bernoulli Type Differential Equations

Theorem 1

Consider a production function $Q:(0,\infty)^2 \to \mathbf{R}_+$, $(K,L) \to Q(K,L)$ homogeneous of first degree, $q(\chi)=Q(\chi,1)$ where $\chi=\frac{K}{L}$ and $y(\chi)=\frac{q'(\chi)}{q(\chi)}$.

1. σ is the elasticity of marginal rate of technical substitution if and only if: $y'(\chi) = \frac{\sigma}{\chi}y(\chi) - (\sigma+1)y^2(\chi)$;

2. If ε_K is the elasticity of production in relation to the capital then: $y'(\chi) = \frac{\varepsilon_K'(\chi)}{\varepsilon_K(\chi)}y(\chi) - \frac{1}{\varepsilon_K(\chi)}y^2(\chi)$. Conversely, if $y'(\chi) = \frac{f'(\chi)}{f(\chi)}y(\chi) - \frac{1}{f(\chi)}y^2(\chi)$ then $\varepsilon_K = f(\chi)$;

3. If ε_L is the elasticity of production in relation to the labor then: $y'(\chi) = -\frac{\varepsilon_L'(\chi)}{1-\varepsilon_L(\chi)}y(\chi) - \frac{1}{1-\varepsilon_L(\chi)}y^2(\chi)$. Conversely, if $y'(\chi) = -\frac{f'(\chi)}{1-f(\chi)}y(\chi) - \frac{1}{1-f(\chi)}y^2(\chi)$ then $\varepsilon_L = f(\chi)$;

4. If $RMS(K,L)=r(\chi)$ is the marginal rate of substitution between L and K then: $y'(\chi) = \left(\frac{r'(\chi)}{r^2(\chi)} - 1\right)y^2(\chi)$. Conversely, if $y'(\chi) = \left(\frac{f'(\chi)}{f^2(\chi)} - 1\right)y^2(\chi)$ with $\lim_{\chi \to 0} f(\chi) = \infty$ then $RMS(K,L)= f(\chi)$;

5. If $RMS(L,K)=r(\chi)$ is the marginal rate of substitution between K and L, then: $y'(\chi) = -(r'(\chi)+1)y^2(\chi)$. Conversely, if: $y'(\chi) = -(f'(\chi)+1)y^2(\chi)$ with $f(0)=0$ then $RMS(L,K)= f(\chi)$.

Proof.

1. From $\sigma=\frac{\chi(y'(\chi)+y^2(\chi))}{y(\chi)(1-\chi y(\chi))}$ follows: $\chi(y'(\chi)+y^2(\chi)) = \sigma y(\chi) - \sigma\chi y^2(\chi)$ from where:

$y'(\chi) = \frac{\sigma}{\chi}y(\chi) - (\sigma+1)y^2(\chi)$. Conversely, if: $y'(\chi) = \frac{\alpha}{\chi}y(\chi) - (\alpha+1)y^2(\chi)$ then:

$$\begin{cases} q'(\chi) = q(\chi)y(\chi) \\ q''(\chi) = q(\chi)y(\chi)\left(\frac{\alpha}{\chi} - \alpha y(\chi)\right) \end{cases}$$

therefore: $\sigma=\frac{\chi q(\chi)q''(\chi)}{q'(\chi)(q(\chi)-\chi q'(\chi))}=\alpha$.

2. From $\varepsilon_K = \chi y(\chi)$ follows: $y(\chi) = \frac{\varepsilon_K(\chi)}{\chi}$ from where:

$$y'(\chi) = \frac{\varepsilon_K'(\chi)\chi - \varepsilon_K(\chi)}{\chi^2} = \frac{\varepsilon_K'(\chi)}{\chi} - \frac{\varepsilon_K(\chi)}{\chi^2} = \frac{\varepsilon_K'(\chi)}{\varepsilon_K(\chi)} y(\chi) - \frac{1}{\varepsilon_K(\chi)} y^2(\chi).$$

Conversely, if: $y'(\chi) = \frac{f'(\chi)}{f(\chi)} y(\chi) - \frac{1}{f(\chi)} y^2(\chi)$ then, for $g(\chi) = \frac{f'(\chi)}{f(\chi)}$, $h(\chi) = -\frac{1}{f(\chi)}$ we have: $\int_0^s g(t)dt = \int_0^s \frac{f'(t)}{f(t)} dt = \ln f(s) - \ln f(0)$ and:

$$y(\chi) = -\frac{e^{\int_0^\chi g(t)dt}}{\int_0^\chi h(s) e^{\int_0^s g(t)dt} ds} = -\frac{e^{\ln\frac{f(\chi)}{f(0)}}}{\int_0^\chi h(s) e^{\ln\frac{f(s)}{f(0)}} ds} = -\frac{\frac{f(\chi)}{f(0)}}{\int_0^\chi h(s)\frac{f(s)}{f(0)} ds} = \frac{f(\chi)}{\int_0^\chi \frac{1}{f(s)} f(s) ds} = \frac{f(\chi)}{\chi}$$

Because $\varepsilon_K = \chi y(\chi)$ follows: $\varepsilon_K = f(\chi)$.

3. How $\varepsilon_L + \varepsilon_K = 1$ follows: $\varepsilon_K'(\chi) = -\varepsilon_L'(\chi)$ therefore, from 2:

$y'(\chi) = -\frac{\varepsilon_L'(\chi)}{1-\varepsilon_L(\chi)} y(\chi) - \frac{1}{1-\varepsilon_L(\chi)} y^2(\chi)$. Conversely, If: $y'(\chi) = -\frac{f'(\chi)}{1-f(\chi)} y(\chi) - \frac{1}{1-f(\chi)} y^2(\chi)$ then, how $g(\chi) = -\frac{f'(\chi)}{1-f(\chi)}$, $h(\chi) = -\frac{1}{1-f(\chi)}$ we have:

$$\int_0^s g(t)dt = -\int_0^s \frac{f'(t)}{1-f(t)} dt = \int_0^s \frac{(1-f(t))'}{1-f(t)} dt = \ln\frac{1-f(s)}{1-f(0)} \text{ and:}$$

$$y(\chi) = -\frac{e^{\ln\frac{1-f(\chi)}{1-f(0)}}}{\int_0^\chi h(s) e^{\ln\frac{1-f(s)}{1-f(0)}} ds} = -\frac{\frac{1-f(\chi)}{1-f(0)}}{\int_0^\chi h(s)\frac{1-f(s)}{1-f(0)} ds} = \frac{1-f(\chi)}{\int_0^\chi \frac{1}{1-f(s)}(1-f(s))ds} = \frac{1-f(\chi)}{\chi}.$$

We have but: $\varepsilon_L = 1 - \chi y(\chi) = f(\chi)$.

4. From $r = RMS(K,L) = \frac{y(\chi)}{1-\chi y(\chi)}$ follows: $y(\chi) = \frac{r(\chi)}{\chi r(\chi)+1}$. We have:

$$y'(\chi) = \frac{r'(\chi)(\chi r(\chi)+1) - r(\chi)(r(\chi)+\chi r'(\chi))}{(\chi r(\chi)+1)^2} = \frac{r'(\chi) - r^2(\chi)}{(\chi r(\chi)+1)^2} = \left(\frac{r'(\chi)}{r^2(\chi)} - 1\right) y^2(\chi)$$

Conversely, if: $y'(\chi) = \left(\frac{f'(\chi)}{f^2(\chi)} - 1\right) y^2(\chi)$, because $g(\chi)=0$, $h(\chi) = \frac{f'(\chi)}{f^2(\chi)} - 1$ we have: $\int_0^s g(t)dt = 0$, and:

$$y(\chi) = -\frac{e^{\int_0^\chi g(t)dt}}{\int_0^\chi h(s) e^{\int_0^s g(t)dt} ds} = -\frac{1}{\int_0^\chi h(s)ds} = -\frac{1}{\int_0^\chi \left(\frac{f'(s)}{f^2(s)} - 1\right) ds} = \frac{1}{\frac{1}{f(\chi)} + \chi}.$$

But RMS(K,L)= $\frac{y(\chi)}{1-\chi y(\chi)} = \frac{\frac{1}{\frac{1}{f(\chi)}+\chi}}{1-\chi\frac{1}{\frac{1}{f(\chi)}+\chi}}$ =f(χ).

5. From r=RMS(L,K)= $\frac{1}{y(\chi)}-\chi$ follows: $y(\chi)=\frac{1}{r(\chi)+\chi}$. We have: $y'(\chi)=-\frac{r'(\chi)+1}{(r(\chi)+\chi)^2}=$ $-\frac{r'(\chi)}{(r(\chi)+\chi)^2}-\frac{1}{(r(\chi)+\chi)^2}=-(r'(\chi)+1)y^2(\chi)$.

Conversely, if: $y'(\chi)=-(f'(\chi)+1)y^2(\chi)$, we have: g(χ)=0, h(χ)= $-(f'(\chi)+1)$ from where:

$\int_0^s g(t)dt$ =0 and: $y(\chi)=-\frac{e^{\int_0^\chi g(t)dt}}{\int_0^\chi h(s)e^{\int_0^s g(t)dt}ds}=-\frac{1}{\int_0^\chi h(s)ds}=\frac{1}{\int_0^\chi (f'(s)+1)ds}=\frac{1}{f(\chi)+\chi}$. But RMS(L,K)= $\frac{1}{y(\chi)}-\chi$ =f(χ).

Q.E.D.

Theorem 2

Consider a production function Q:$(0,\infty)^2\rightarrow\mathbf{R}_+$, (K,L)→Q(K,L) homogeneous of first degree, q(χ)=Q(χ,1) where χ= $\frac{K}{L}$ and y(χ)= $\frac{q'(\chi)}{q(\chi)}$.

1. If σ is the elasticity of marginal rate of technical substitution then ∃ g,h:(0,∞)→**R**, continous on (0,∞), χg(χ)=σ, h(χ)=-σ-1 such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. Conversely, if ∃ g,h:(0,∞)→**R**, continous on (0,∞), h(χ)=-χg(χ)-1 such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$ then: σ=χg(χ).

2. If ε_K is the elasticity of production in relation to the capital then ∃ g,h:(0,∞)→**R**, continous on (0,∞), g(χ)= $\frac{\varepsilon_K'(\chi)}{\varepsilon_K(\chi)}$, h(χ)= $-\frac{1}{\varepsilon_K(\chi)}$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. Conversely, if ∃ g,h:(0,∞)→**R**, continous on (0,∞), $g(\chi)=-|(\ln h(\chi))|'$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$ then: $\varepsilon_K=-\frac{1}{h(\chi)}$.

3. If ε_L is the elasticity of production in relation to the labor then ∃ g,h:(0,∞)→**R**, continous on (0,∞), g(χ)= $-\frac{\varepsilon_L'(\chi)}{1-\varepsilon_L(\chi)}$, h(χ)= $-\frac{1}{1-\varepsilon_L(\chi)}$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. Conversely, if ∃ g,h:(0,∞)→**R**, continous on (0,∞), $g(\chi)=-(\ln|h(\chi)|)'$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$ then: $\varepsilon_L=1+\frac{1}{h(\chi)}$.

4. If RMS(K,L)=r(χ) is the marginal rate of substitution between L and K then ∃ h:(0,∞)→**R**, continous on (0,∞), $h(\chi)=\frac{r'(\chi)}{r^2(\chi)}-1$ such that $y'(\chi)=h(\chi)y^2(\chi)$. Conversely, if ∃ h:(0,∞)→**R**, continous on (0,∞), such that $y'(\chi)=h(\chi)y^2(\chi)$ then: $RMS(K,L)=-\frac{1}{\chi+\int_0^{\chi}h(s)ds}$.

5. If RMS(L,K)=r(χ) is the marginal rate of substitution between K and L, then ∃ h:(0,∞)→**R**, continous on (0,∞), $h(\chi)=-(r'(\chi)+1)$ such that $y'(\chi)=h(\chi)y^2(\chi)$. Conversely, if ∃ h:(0,∞)→**R**, continous on (0,∞), such that $y'(\chi)=h(\chi)y^2(\chi)$ then: $RMS(L,K)=-\int_0^{\chi}h(s)ds-\chi$.

Proof.

1. From the theorem 1, considering $g(\chi)=\frac{\sigma}{\chi}$ and h(χ)=-σ-1 the first part of the assertion is proved.

Suppose now that there are g,h:(0,∞)→**R**, continous on (0,∞), h(χ)=-χg(χ)-1 such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. We have:

$$\sigma=\frac{\chi(y'(\chi)+y^2(\chi))}{y(\chi)(1-\chi y(\chi))}=\frac{\chi(g(\chi)y(\chi)+h(\chi)y^2(\chi)+y^2(\chi))}{y(\chi)(1-\chi y(\chi))}=\frac{\chi(g(\chi)y(\chi)-\chi g(\chi)y^2(\chi))}{y(\chi)(1-\chi y(\chi))}=$$
$$\frac{\chi g(\chi)y(\chi)(1-\chi y(\chi))}{y(\chi)(1-\chi y(\chi))}=\chi g(\chi).$$

2. From theorem 1, considering $g(\chi)=\frac{\varepsilon_K'(\chi)}{\varepsilon_K(\chi)}$, $h(\chi)=-\frac{1}{\varepsilon_K(\chi)}$ follows the first implication. Suppose now that there are g,h:(0,∞)→**R**, continous on (0,∞), $g(\chi)=-(\ln h(\chi))'$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. We have: : $\int_0^s g(t)dt=-\int_0^s(\ln|h(t)|)'dt=\ln\left|\frac{h(0)}{h(s)}\right|$ from where:

$$y(\chi)=-\frac{e^{\int_0^{\chi}g(t)dt}}{\int_0^{\chi}h(s)e^{\int_0^{s}g(t)dt}ds}=-\frac{\left|\frac{h(0)}{h(\chi)}\right|}{\int_0^{\chi}h(s)\left|\frac{h(0)}{h(s)}\right|ds}=-\frac{1}{\chi h(\chi)}$$

. Therefore: $\varepsilon_K=\chi y(\chi)=-\frac{1}{h(\chi)}$.

3. From theorem 1, considering $g(\chi)=-\frac{\varepsilon_L'(\chi)}{1-\varepsilon_L(\chi)}$, $h(\chi)=-\frac{1}{1-\varepsilon_L(\chi)}$ follows the first implication. Suppose now that there are g,h:(0,∞)→**R**, continous on (0,∞), $g(\chi)=-(\ln h(\chi))'$ such that $y'(\chi)=g(\chi)y(\chi)+h(\chi)y^2(\chi)$. We have: : $\int_0^s g(t)dt=-\int_0^s(\ln|h(t)|)'dt=\ln\left|\frac{h(0)}{h(s)}\right|$ from where:

$$y(\chi)=-\frac{e^{\int_0^{\chi}g(t)dt}}{\int_0^{\chi}h(s)e^{\int_0^{s}g(t)dt}ds}=-\frac{\left|\frac{h(0)}{h(\chi)}\right|}{\int_0^{\chi}h(s)\left|\frac{h(0)}{h(s)}\right|ds}=-\frac{1}{\chi h(\chi)}$$

. Therefore: $\varepsilon_L=1-\chi y(\chi);=1+\frac{1}{h(\chi)}$.

4. From theorem 1, considering $h(\chi)=\frac{r'(\chi)}{r^2(\chi)}-1$ follows the first assertion. Suppose now that there is $h:(0,\infty)\to\mathbf{R}$, continous on $(0,\infty)$ such that $y'(\chi)=h(\chi)y^2(\chi)$. We have: $\frac{dy}{y^2}=h(\chi)d\chi$ from where $\frac{1}{y}=-\int_0^\chi h(s)ds$ therefore: $y(\chi)=-\frac{1}{\int_0^\chi h(s)ds}$. We have but: $RMS(K,L)=\frac{y(\chi)}{1-\chi y(\chi)}=\frac{-\frac{1}{\int_0^\chi h(s)ds}}{1+\frac{\chi}{\int_0^\chi h(s)ds}}=-\frac{1}{\chi+\int_0^\chi h(s)ds}$.

5. From theorem 1, considering $h(\chi)=-(r'(\chi)+1)$ follows the first assertion. uppose now that there is $h:(0,\infty)\to\mathbf{R}$, continous on $(0,\infty)$ such that $y'(\chi)=h(\chi)y^2(\chi)$. We obtain finally: $y(\chi)=-\frac{1}{\int_0^\chi h(s)ds}$. We have but: $RMS(L,K)=\frac{1}{y(\chi)}-\chi=-\int_0^\chi h(s)ds-\chi$.

Q.E.D.

4. Application

Considering the Cobb-Douglas function: $Q(K,L)=AK^{\alpha}L^{1-\alpha}$, $\alpha\in(0,1)$ we have: $q(\chi)=Q(\chi,1)=A\chi^{\alpha}$, $q'(\chi)=A\alpha\chi^{\alpha-1}$ from where: $y(\chi)=\frac{q'(\chi)}{q(\chi)}=\frac{\alpha}{\chi}$, $y'(\chi)=-\frac{\alpha}{\chi^2}=-\frac{1}{\chi}y(\chi)$. From theorem 2, the functions $g(\chi)=-\frac{1}{\chi}$ and $h(\chi)=0$ satisfying the hypotesis of the first assertion, therefore: $\sigma=\chi g(\chi)=-1$. Also, we can write: $y'(\chi)=-\frac{1}{\alpha}y^2(\chi)$. From theorem 2.2) follows that $g(\chi)=0$ and $h(\chi)=-\frac{1}{\alpha}$ satisfying the condition: $g(\chi)=-|(\ln h(\chi))|'$ therefore: $\varepsilon_K=-\frac{1}{h(\chi)}=\alpha$. Analogously, $\varepsilon_L=1-\alpha$. From theorem 2.4) we have for $h(\chi)=-\frac{1}{\alpha}$: $RMS(K,L)=-\frac{1}{\chi+\int_0^\chi h(s)ds}=-\frac{1}{\chi-\int_0^\chi \frac{1}{\alpha}ds}=-\frac{1}{\chi-\frac{\chi}{\alpha}}=\frac{\alpha}{(1-\alpha)\chi}$. Analogously: $RMS(L,K)=-\int_0^\chi h(s)ds-\chi=\int_0^\chi \frac{1}{\alpha}ds-\chi=\frac{(1-\alpha)\chi}{\alpha}$.

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