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The Cournot Equilibrium for n Firms

Gina Ioan[1], Catalin Angelo Ioan[2]

Abstract: Oligopoly is a market situation where there are a small number of bidders (at least two) of a good non-substituent and a sufficient number of consumers. The paper analyses the Cournot equilibrium in both cases where each firm assumes the role of leadership and after when firms act simultaneously on market. There are obtained the equilibrium productions, maximum profits and sales price.

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1 Introduction

For Homo Oeconomicus (the abstract concept around which Adam Smith builds scientific edifice), the ultimate goal is to achieve a maximum level of utility or profit with a minimum of effort. Regardless of the time, circumstances, or players, maximizing the effects can only get in a perfectly competitive scene, where responsibility of a efficient allocation of resources in a market have income, prices and profits, guided by an "invisible hand" or the freedom to choose.

In this scene, in which resists the strongest stands up to and where competition leads to efficiency and progress, the rules are quite tough, based both on their own actions and the anticipate the actions of other players.. The game, in its dynamics may be perceived as uninteresting to the player on stage (because he seeks only purpose), but it can also be interesting for the same player who withdraws from the scene and analyze it from the perspective of the researcher economist.

Depending on the number, the economic power of economic agents (either producers or consumers), the price elasticity of demand, the degree of mobility of factors of production, the market can be divided into: perfectly competitive market, a model more theoretical and imperfect competitive market where we meet market monopoly, monopolistic competition market and oligopolistic market, the latter manifesting the actual scene of the market economy.

When we refer to a perfectly competitive market, the price can not be influenced, it is resulting from the free play of supply and demand, the manufacturer aiming to maximize the production or cost minimization knowing the identity $R_{mg}=C_{mg}=price$ maximizes profit. If we discuss monopolistic

[1] Assistant professor, PhD, "Danubius" University of Galati, Romania, Address: 3 Galati Boulevard, 800654 Galati, Romania, Tel.: +40.372.361.102, fax: +40.372.361.290, E-mail: ginaioan@univ-danubius.ro.

[2] Associate Professor, PhD, "Danubius" University of Galati, Romania, Address: 3 Galati Boulevard, 800654 Galati, Romania, Tel.: +40.372.361.102, fax: +40.372.361.290, Corresponding author: catalin_angelo_ioan@univ-danubius.ro.

competition market, the monopolist is protected by barriers (natural and legal) to the entry of competitors on the market, and he can maximize profits at a price determined by him.

Regarding oligopolistic market (duopoly) uncooperative, the question which is also subject to this scientific approach is how does oligopolistic / duopolistic to maximize profit knowing that its decision-making strategy depends on the behavior of other players, meaning a strategic interaction occurs.
To answer the question above, we report on game theory, Cournot equilibrium and Nash equilibrium.

Elements of game theory provides us the tools required to formulate, analyze, structure and understand the scenarios arising from strategic interaction.

A major contribution to game theory had John Nash, whose research is based on the absence of cooperation assuming that each player acts independently without cooperation with other players.

Nash equilibrium is a holistic strategy and involves two or more players in which it is assumed that each of them predicts the equilibrium strategy of the other players and no player wins nothing by changing their strategy. Games can be with unique and optimal Nash equilibrium and also may be games that allowed several Nash equilibria.

Cournot duopoly is characterized by symmetric role of companies, each agreeing to have the same role in the market. The profit of each firm depends on the amount produced by the other company. The set of options is optimal and it is the Cournot equilibrium occurs when maximizing profits for a given level of production of competitor player. In this reaserch it is studied the delay Cournot duopoly and dynamical behaviors of the game. The decision taken by a player at *t+1* moment depends on the decision taken by the other player at *t* moment, this making decisions more relevant.

2 The Cournot Equilibrium for Oligopoly

Let consider, for the beginning m firms F_i, $i=\overline{1,m}$ with price function: $p(Q)=a-bQ$, $a,b>0$ and the total cost of production being $TC_i=\alpha_i Q$, $i=\overline{1,m}$.

We assume that each firm assumes, successive a leading role. Consider therefore the reaction function of the company F_i to the others at a moment t:

$$Q_{i,t}= f_i\left(Q_{1,t-1},...,\hat{Q}_{i,t-1},...,Q_{m,t-1}\right)\ \forall t\geq 1,\ i=\overline{1,m}$$

where ^ means that the term missing.

If $\exists \lim_{t\to\infty} Q_{i,t} = Q_i^*\ \forall i=\overline{1,m}$ we shall say that the vector in $\mathbf{R}^m$: $\left(Q_1^*,...,Q_m^*\right)$ is a Cournot equilibrium.

We ask ourselves whether this equilibrium exists, and if so, which yields the equilibrium productions of the m firms.

Considering a fixed time t+1, the selling price of the products of a company F_i, which assumes a leadership role is determined both by its production at this moment, as well as other production companies at the previous time t when the company F_i has informations about the competitors.

Therefore, the selling price will be:

$$p\left(\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} + Q_{i,t+1}\right) = a - b\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} - bQ_{i,t+1}$$

At time t+1, the F_i firm's profit is, for a quantity $Q_{i,t+1}$:

$$\Pi_{i,t+1} = p\left(\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} + Q_{i,t+1}\right)Q_{i,t+1} - TC_{i,t+1} = \left(a - b\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} - bQ_{i,t+1}\right)Q_{i,t+1} - \alpha_i Q_{i,t+1} \text{ from where:}$$

$$\Pi_{i,t+1} = -bQ_{i,t+1}^2 + \left(a - b\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} - \alpha_i\right)Q_{i,t+1}$$

The maximization of the F_i's profit at time t+1 returns to the cancellation of partial derivative of profit with respect to its production at that time, so: $\frac{\partial \Pi_{i,t+1}}{\partial Q_{i,t+1}} = -2bQ_{i,t+1} + \left(a - b\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t} - \alpha_i\right) = 0$ from where:

$$Q_{i,t+1} = \frac{a-\alpha_i}{2b} - \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{m} Q_{k,t}\ ,\ i=\overline{1,m}$$

In a matricial expression the above relations can be written as:

$$\begin{pmatrix} Q_{1,t+1} \\ \dots \\ Q_{i,t+1} \\ \dots \\ Q_{m,t+1} \end{pmatrix} = \begin{pmatrix} 0 & \dots & -\frac{1}{2} & \dots & -\frac{1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{2} & \dots & 0 & \dots & -\frac{1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{2} & \dots & -\frac{1}{2} & \dots & 0 \end{pmatrix} \begin{pmatrix} Q_{1,t} \\ \dots \\ Q_{i,t} \\ \dots \\ Q_{m,t} \end{pmatrix} + \begin{pmatrix} \frac{a-\alpha_1}{2b} \\ \dots \\ \frac{a-\alpha_i}{2b} \\ \dots \\ \frac{a-\alpha_m}{2b} \end{pmatrix}$$

Noting now for simplicity:

$$A = \begin{pmatrix} 0 & \dots & -\frac{1}{2} & \dots & -\frac{1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{2} & \dots & 0 & \dots & -\frac{1}{2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{2} & \dots & -\frac{1}{2} & \dots & 0 \end{pmatrix} = -\frac{1}{2}D + \frac{1}{2}I_m \text{ where } D = \begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & 1 & \dots & 1 \end{pmatrix},$$

$$Q_t = \begin{pmatrix} Q_{1,t} & \dots & Q_{i,t} & \dots & Q_{m,t} \end{pmatrix}^t,\ C = \begin{pmatrix} \frac{a-\alpha_1}{2b} & \dots & \frac{a-\alpha_i}{2b} & \dots & \frac{a-\alpha_m}{2b} \end{pmatrix}^t$$

we find that:

$$Q_{t+1} = AQ_t + C,\ t \geq 0$$

Let now P(n): $Q_{t+n}=A^nQ_t+(A^{n-1}+...+A+I_m)C$, $n\geq 1$ (*where I_m is the unit matrix of order m*) – the vector of the production of the m firms at a point t+n with n units of time offset to a reference time t.

How P(1): $Q_{t+1}=AQ_t+C$ is true, suppose that P(n) is true. We have: P(n+1): $Q_{t+n+1}=AQ_{t+n}+C=A(A^nQ_t+(A^{n-1}+...+A+I_m)C)+C=A^{n+1}Q_t+(A^n+...+A+I_m)C$ – true, therefore, we have proved by mathematical induction that:

$$Q_{t+n}=A^nQ_t+(A^{n-1}+...+A+I_m)C,\ \forall n\geq 0\ \forall t\geq 0$$

In particular, for t=0 (reference to the beginning of the m firms), we obtain:

$$Q_n=A^nQ_0+(A^{n-1}+...+A+I_m)C$$

We ask ourselves, naturally the problem of determining the vector Q_n.

Let us note first that the matrix $A-I_m$ is invertible and:

$$(A-I_m)^{-1}=\frac{1}{m+1}\begin{pmatrix} -2m & ... & 2 & ... & 2 \\ ... & ... & ... & ... & ... \\ 2 & ... & -2m & ... & 2 \\ ... & ... & ... & ... & ... \\ 2 & ... & 2 & ... & -2m \end{pmatrix}=\frac{2}{m+1}D-2I_m$$

For the sum: $A^{n-1}+...+A+I_m$ we will take into account that: $(A-I_m)(A^{n-1}+...+A+I_m)=A^n+...+A^2+A-A^{n-1}-...-A-I_m=A^n-I_m$ therefore:

$$A^{n-1}+...+A+I_m=(A-I_m)^{-1}(A^n-I_m)$$

But we have: $A^n=\left(-\frac{1}{2}\right)^n(D-I_m)^n=\left(-\frac{1}{2}\right)^n\sum_{s=0}^{n}(-1)^s\binom{n}{s}D^{n-s}$ where $\binom{n}{s}$ is the number of combinations of n items taken by s. How $D^{n-s}=m^{n-s-1}D$ for $n-s\geq 1$ follows:

$$A^n=\frac{1}{2^n}\left(I_m+\frac{(1-m)^n-1}{m}D\right)$$

We therefore have:

$$A^{n-1}+...+A+I_m=(A-I_m)^{-1}(A^n-I_m)=\left(\frac{m+1-2^n m-(1-m)^n}{2^{n-1}m(m+1)}\right)D-2\left(\frac{1}{2^n}-1\right)I_m$$

from where:

$$Q_n=A^nQ_0+(A^{n-1}+...+A+I_m)C=\frac{1}{2^n}Q_0+\frac{(1-m)^n-1}{2^n m}DQ_0+\frac{m+1-2^n m-(1-m)^n}{2^{n-1}m(m+1)}DC-2\left(\frac{1}{2^n}-1\right)C$$

or, in other terms:

$$\begin{pmatrix} Q_{1,n} & ... & Q_{i,n} & ... & Q_{m,n} \end{pmatrix}^t=$$

$$\frac{1}{2^n}\left(Q_{1,0} \quad \dots \quad Q_{i,0} \quad \dots \quad Q_{m,0}\right)^t + \frac{(1-m)^n - 1}{2^n m}\left(\sum_{k=1}^{m} Q_{k,0} \quad \dots \quad \sum_{k=1}^{m} Q_{k,0} \quad \dots \quad \sum_{k=1}^{m} Q_{k,0}\right)^t +$$

$$\frac{m+1-2^n m-(1-m)^n}{2^{n-1} m(m+1)}\left(\frac{ma-\sum_{k=1}^{m}\alpha_k}{2b} \quad \dots \quad \frac{ma-\sum_{k=1}^{m}\alpha_k}{2b} \quad \dots \quad \frac{ma-\sum_{k=1}^{m}\alpha_k}{2b}\right)^t -$$

$$2\left(\frac{1}{2^n}-1\right)\left(\frac{a-\alpha_1}{2b} \quad \dots \quad \frac{a-\alpha_i}{2b} \quad \dots \quad \frac{a-\alpha_m}{2b}\right)^t$$

from where:

$$Q_{i,n} = \frac{1}{2^n} Q_{i,0} + \frac{(1-m)^n - 1}{2^n m}\sum_{k=1}^{m} Q_{k,0} + \frac{\left(m+1-2^n m-(1-m)^n\right)\left(ma-\sum_{k=1}^{m}\alpha_k\right)}{2^n bm(m+1)} - 2\left(\frac{1}{2^n}-1\right)\frac{a-\alpha_i}{2b}$$

Because: $\lim_{n\to\infty}\frac{1}{2^n}=0$, $\lim_{n\to\infty}\frac{(1-m)^n-1}{2^n m}=\frac{1}{m}\lim_{n\to\infty}\left(\frac{1-m}{2}\right)^n=\begin{cases}0 \text{ if } m=2\\ \text{not exists if } m>2\end{cases}$,

$$\lim_{n\to\infty}\frac{m+1-2^n m-(1-m)^n}{2^{n-1}m(m+1)} = -\frac{2}{m+1}-\frac{2}{m(m+1)}\lim_{n\to\infty}\left(\frac{1-m}{2}\right)^n=\begin{cases}-\frac{2}{3} \text{ if } m=2\\ \text{not exists if } m>2\end{cases}, \quad \lim_{n\to\infty}2\left(\frac{1}{2^n}-1\right)=-2$$

we finally have that:

$$\lim_{n\to\infty} Q_n = \begin{cases}-\frac{2}{3}DC+2C=\begin{pmatrix}\frac{a-2\alpha_1+\alpha_2}{3b}\\ \frac{a+\alpha_1-2\alpha_2}{3b}\end{pmatrix} \text{ if } m=2\\ \text{not exists if } m>2\end{cases}$$

We determined so that the process has limit if and only if m=2 so there is a duopoly, in which case we have:

$$Q_1^* = \frac{a-2\alpha_1+\alpha_2}{3b}, \quad Q_2^* = \frac{a+\alpha_1-2\alpha_2}{3b}$$

$$p^* = a-b\,(Q_A^*+Q_B^*) = \frac{a+\alpha_1+\alpha_2}{3}.$$

$$\Pi_1^* = \frac{(a-2\alpha_1+\alpha_2)^2}{9b}, \quad \Pi_2^* = \frac{(a+\alpha_1-2\alpha_2)^2}{9b}$$

Computing the difference: $\Pi_1^*-\Pi_2^*=\frac{(a-2\alpha_1+\alpha_2)^2}{9b}-\frac{(a+\alpha_1-2\alpha_2)^2}{9b}$ we get:

$$\Pi_1^*-\Pi_2^* = \frac{(a-2\alpha_1+\alpha_2)^2}{9b}\left(1-\left(\frac{a+\alpha_1-2\alpha_2}{a-2\alpha_1+\alpha_2}\right)^2\right)$$

The requirement to $\Pi_1^*>\Pi_2^*$ returns to: $\frac{a+\alpha_1-2\alpha_2}{a-2\alpha_1+\alpha_2}\in(-1,1)$ which is equivalent to:

$$\begin{cases} \dfrac{2a-\alpha_1-\alpha_2}{a-2\alpha_1+\alpha_2} > 0 \\ \dfrac{3(\alpha_1-\alpha_2)}{a-2\alpha_1+\alpha_2} < 0 \end{cases}$$

If $a > 2\alpha_1 - \alpha_2$ the conditions become: $\begin{cases} \alpha_1+\alpha_2 < 2a \\ \alpha_1 < \alpha_2 \end{cases}$ and if $a < 2\alpha_1 - \alpha_2$ to $\begin{cases} \alpha_1+\alpha_2 > 2a \\ \alpha_1 > \alpha_2 \end{cases}$.

In order to $\Pi_2^* > \Pi_1^*$ from the same set of conditions, we have: $\dfrac{a+\alpha_1-2\alpha_2}{a-2\alpha_1+\alpha_2} \in (-\infty,-1)\cup(1,\infty)$ that is:

$$\begin{cases} \dfrac{2a-\alpha_1-\alpha_2}{a-2\alpha_1+\alpha_2} < 0 \\ \text{or} \\ \dfrac{3(\alpha_1-\alpha_2)}{a-2\alpha_1+\alpha_2} > 0 \end{cases}$$

If $a > 2\alpha_1 - \alpha_2$ the conditions become: $\begin{cases} \alpha_1+\alpha_2 > 2a \\ \text{or} \\ \alpha_1 > \alpha_2 \end{cases}$, and if $a < 2\alpha_1 - \alpha_2$ to $\begin{cases} \alpha_1+\alpha_2 < 2a \\ \text{or} \\ \alpha_1 < \alpha_2 \end{cases}$.

Considering a system of axes $O\alpha_1\alpha_2$ it follows that in the areas 1 and 2, the first company profit will be strictly greater than that of the second company, and in the third region, the second company profit will be strictly higher than that of the first company (figure 1).

If $a = 2\alpha_1 - \alpha_2$ then: Π_1^*=0 and how Π_2^*>0 follows, obviously that $\Pi_1^* < \Pi_2^*$. Similarly, if: $a = 2\alpha_2 - \alpha_1$ then: Π_2^*=0 and how Π_1^*>0 follows that $\Pi_1^* > \Pi_2^*$.

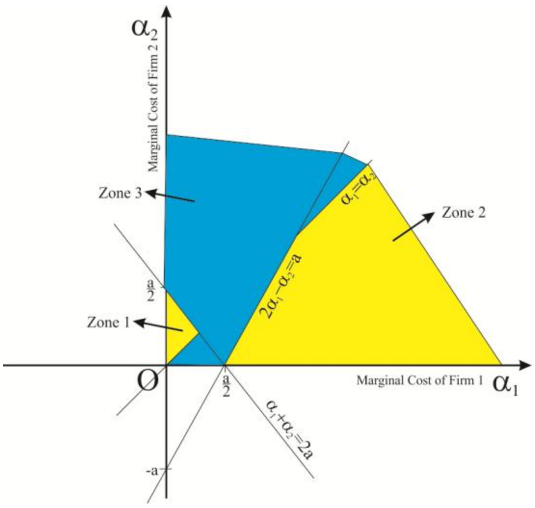

Figure 1

3 The Cournot-Nash Equilibrium

Considering again the m firms above, we assume now that the firms act independently, the selling price being the same for all firms, depending only on the total production.

The profit function for the company F_k is therefore:

$$\Pi_k(Q_1,...,Q_n)=p\left(\sum_{i=1}^{m}Q_i\right)Q_k-\alpha_k Q_k\text{ , k}=\overline{1,m}$$

The condition of profit maximization implies:

$$\frac{\partial\Pi_k}{\partial Q_k}=p'\left(\sum_{i=1}^{m}Q_i\right)Q_k+p\left(\sum_{i=1}^{m}Q_i\right)-\alpha_k=0$$

which is equivalent to:

$$Q_k+\sum_{i=1}^{m}Q_i=\frac{a-\alpha_k}{b}\text{ , k}=\overline{1,m}$$

Adding the m relations follows: $\sum_{i=1}^{m}Q_i=\frac{ma-\sum_{i=1}^{m}\alpha_i}{(m+1)b}$ from where, finally:

$$Q_k^*=\frac{a-(m+1)\alpha_k+\sum_{i=1}^{m}\alpha_i}{(m+1)b}\text{ , k}=\overline{1,m}$$

The total price is:

$$p^*=p\left(\sum_{k=1}^{m}Q_k^*\right)=\frac{a+\sum_{i=1}^{m}\alpha_i}{m+1}$$

and the profits of companies:

$$\Pi_k(Q_1^*,...,Q_m^*)=\left(\frac{a+\sum_{i=1}^{m}\alpha_i}{m+1}-\alpha_k\right)Q_k^*=\frac{\left(a-(m+1)\alpha_k+\sum_{i=1}^{m}\alpha_i\right)^2}{(m+1)^2 b}$$

For m=2 we have:

$$Q_1^*=\frac{a-2\alpha_1+\alpha_2}{3b}\text{ , }Q_2^*=\frac{a-2\alpha_2+\alpha_1}{3b}$$

total price: $p^*=\frac{a+\alpha_1+\alpha_2}{3}$ and firms profits:

$$\Pi_1(Q_1^*,Q_2^*)=\frac{(a-2\alpha_1+\alpha_2)^2}{9b}\text{ , }\Pi_2(Q_1^*,Q_2^*)=\frac{(a-2\alpha_2+\alpha_1)^2}{9b}$$

It is noted that at equilibrium the two companies will have the same optimal amounts and record the same profits as if taking in the leadership analysis.

Comparing with the first case, when at the time n for two companies (m=2), we have:

$$Q_{1,n}=\frac{1+(-1)^n}{2^{n+1}}Q_{1,0}+\frac{(-1)^n-1}{2^{n+1}}Q_{2,0}+\frac{2\left(2^n+(-1)^{n+1}\right)a+\left(3-2^{n+2}+(-1)^n\right)\alpha_1-\left(3-2^{n+1}+(-1)^{n+1}\right)\alpha_2}{3\cdot 2^{n+1}b}$$

$$Q_{2,n}=\frac{(-1)^n-1}{2^{n+1}}Q_{1,0}+\frac{1+(-1)^n}{2^{n+1}}Q_{2,0}+\frac{2\left(2^n+(-1)^{n+1}\right)a-\left(3-2^{n+1}+(-1)^{n+1}\right)\alpha_1+\left(3-2^{n+2}+(-1)^n\right)\alpha_2}{3\cdot 2^{n+1}b}$$

we obtain:

$$Q_{1,n}-Q_1^*=\frac{1+(-1)^n}{2^{n+1}}Q_{1,0}+\frac{(-1)^n-1}{2^{n+1}}Q_{2,0}+\frac{2(-1)^{n+1}a+\left(3+(-1)^n\right)\alpha_1-\left(3+(-1)^{n+1}\right)\alpha_2}{3\cdot 2^{n+1}b}$$

$$Q_{2,n}-Q_2^*=\frac{(-1)^n-1}{2^{n+1}}Q_{1,0}+\frac{1+(-1)^n}{2^{n+1}}Q_{2,0}+\frac{2(-1)^{n+1}a-\left(3+(-1)^{n+1}\right)\alpha_1+\left(3+(-1)^n\right)\alpha_2}{3\cdot 2^{n+1}b}$$

As a result, for n=odd, we have:

$$Q_{1,n}-Q_1^*=\frac{a+\alpha_1-2\alpha_2-3bQ_{2,0}}{3\cdot 2^n b},\ Q_{2,n}-Q_2^*=\frac{a-2\alpha_1+\alpha_2-3bQ_{1,0}}{3\cdot 2^n b}$$

and for n=even:

$$Q_{1,n}-Q_1^*=-\frac{a-2\alpha_1+\alpha_2-3bQ_{1,0}}{3\cdot 2^n b},\ Q_{2,n}-Q_2^*=-\frac{a+\alpha_1-2\alpha_2-3bQ_{2,0}}{3\cdot 2^n b}$$

From the above, the following conditions are observed:

Case 1: If $Q_{1,0}<\frac{a-2\alpha_1+\alpha_2}{3b}$, $Q_{2,0}<\frac{a+\alpha_1-2\alpha_2}{3b}$ then at odd moments $Q_{1,n}>Q_1^*$, $Q_{2,n}>Q_2^*$, and then at even moments: $Q_{1,n}<Q_1^*$, $Q_{2,n}<Q_2^*$;

Case 2: If $Q_{1,0}<\frac{a-2\alpha_1+\alpha_2}{3b}$, $Q_{2,0}>\frac{a+\alpha_1-2\alpha_2}{3b}$ then at odd moments $Q_{1,n}<Q_1^*$, $Q_{2,n}>Q_2^*$, and then at even moments: $Q_{1,n}<Q_1^*$, $Q_{2,n}>Q_2^*$;

Case 3: If $Q_{1,0}>\frac{a-2\alpha_1+\alpha_2}{3b}$, $Q_{2,0}<\frac{a+\alpha_1-2\alpha_2}{3b}$ then at odd moments $Q_{1,n}>Q_1^*$, $Q_{2,n}<Q_2^*$, and then at even moments: $Q_{1,n}>Q_1^*$, $Q_{2,n}<Q_2^*$;

Case 4: If $Q_{1,0}>\frac{a-2\alpha_1+\alpha_2}{3b}$, $Q_{2,0}>\frac{a+\alpha_1-2\alpha_2}{3b}$ then at odd moments $Q_{1,n}<Q_1^*$, $Q_{2,n}<Q_2^*$, and then at even moments: $Q_{1,n}>Q_1^*$, $Q_{2,n}>Q_2^*$

Following these considerations, we see that if all firms assume successively the leadership role their outputs can switch on both sides of the equilibrium (if in the case of the existence of the limit or in the contrary case).

More specifically, however, if the initial yields are sufficiently small (large) compared with $\frac{a-2\alpha_1+\alpha_2}{3b}$ and $\frac{a+\alpha_1-2\alpha_2}{3b}$ respectively, the the production alternates at odd or even moments.

If, however, the initial production of a firm is sufficiently small and the other's large enough, then those with low initial production will be directed upward (at any time) to the equilibrium production, whereas those with higher initial production will down to the equilibriu.

The consequence of this is that the company with a higher initial production will not give leadership to the other, preferring to ignore the actions of others and maintaining a constant production.

Now consider a situation in which a new company F_{m+1} enters on market and having marginal cost $MC=\alpha_{m+1}$. The new equilibriums are:

$$Q^*_{new,k} = \frac{a-(m+2)\alpha_k + \sum_{i=1}^{m}\alpha_i + \alpha_{m+1}}{(m+2)b}, \ k=\overline{1,m}$$

$$Q^*_{new,m+1} = \frac{a-(m+1)\alpha_{m+1} + \sum_{i=1}^{m}\alpha_i}{(m+2)b}$$

If before the entry of the new firm the total ammount was: $Q_{old} = \sum_{k=1}^{m} Q^*_k = \frac{ma - \sum_{i=1}^{m}\alpha_i}{(m+1)b}$ now we have: $Q_{new} = \sum_{k=1}^{m} Q^*_{new,k} + Q^*_{new,m+1} = \frac{(m+1)a - \sum_{i=1}^{m}\alpha_i - \alpha_{m+1}}{(m+2)b}$. The difference between the new and the old ammount is:

$$Q_{new} - Q_{old} = \frac{a + \sum_{i=1}^{m}\alpha_i - (m+1)\alpha_{m+1}}{(m+1)(m+2)b}$$

the total price being: $p^*_{new} = p\left(\sum_{k=1}^{m+1} Q^*_k\right) = \frac{a + \sum_{i=1}^{m}\alpha_i + \alpha_{m+1}}{m+2}$ with the difference:

$$p^*_{new} - p^*_{old} = \frac{a + \sum_{i=1}^{m}\alpha_i + \alpha_{m+1}}{m+2} - \frac{a + \sum_{i=1}^{m}\alpha_i}{m+1} = -b(Q_{new} - Q_{old})$$

Also for the profits:

$$\Pi_{new,k}(Q^*_{new,1},\ldots,Q^*_{new,m+1}) = \frac{\left(a-(m+2)\alpha_k + \sum_{i=1}^{m}\alpha_i + \alpha_{m+1}\right)^2}{(m+2)^2 b}, \ k=\overline{1,m}$$

$$\Pi_{new,m+1}(Q^*_{new,1},\ldots,Q^*_{new,m+1}) = \frac{\left(a-(m+1)\alpha_{m+1} + \sum_{i=1}^{m}\alpha_i\right)^2}{(m+2)^2 b}$$

and:

$$\Pi_{new,k} - \Pi_{old,k} = \frac{1}{(m+1)^2(m+2)^2 b} \cdot$$

$$\left[(m+1)^2\alpha_{m+1}^2 + 2(m+1)^2\left(\left(a+\sum_{i=1}^{m}\alpha_i\right) - (m+2)\alpha_k\right)\alpha_{m+1} - \left(a+\sum_{i=1}^{m}\alpha_i\right)\left((2m+3)\left(a+\sum_{i=1}^{m}\alpha_i\right) - 2(m+1)(m+2)\alpha_k\right)\right]$$

Considering the equation of degree II in α_{m+1} we have its roots:

$$\alpha'_{m+1} = \frac{a+\sum_{i=1}^{m}\alpha_i}{m+1},\ \alpha''_{m+1} = \frac{(2m^2+4m+1)\alpha_k - (2m+3)\left(a+\sum_{\substack{i=1\\ i\neq k}}^{m}\alpha_i\right)}{(m+1)},\ \alpha'_{m+1} < \alpha''_{m+1}.$$

In order that: $\Pi_{new,k} > \Pi_{old,k}$ we must have that:

$$\alpha_{m+1} \in (-\infty, \alpha'_{m+1}) \cup (\alpha''_{m+1}, \infty) = \left(-\infty, \frac{a+\sum_{i=1}^{m}\alpha_i}{m+1}\right) \cup \left(\frac{2(m+1)(m+2)\alpha_k - (2m+3)\left(a+\sum_{i=1}^{m}\alpha_i\right)}{(m+1)}, \infty\right)$$

<u>Case 1</u>

$$\alpha_{m+1} < \frac{a+\sum_{i=1}^{m}\alpha_i}{m+1} \Rightarrow Q_{new} > Q_{old},\ p^*_{new} < p^*_{old},\ \Pi_{new,k} > \Pi_{old,k}\ \ \forall k=\overline{1,m}$$

so if the new company will have a marginal cost small enough, then the total output and profits of the old firms will grow, the selling price dropping.

<u>Case 2</u>

$$\alpha_{m+1} > \frac{a+\sum_{i=1}^{m}\alpha_i}{m+1} \Rightarrow Q_{new} < Q_{old},\ p^*_{new} > p^*_{old}, \text{ and}$$

$$\Pi_{new,k} > \Pi_{old,k} \text{ if } \alpha_k < \frac{(2m+3)\left(a+\sum_{\substack{i=1\\ i\neq k}}^{m}\alpha_i\right) + (m+1)\alpha_{m+1}}{2m^2+4m+1} \text{ and}$$

$$\Pi_{new,k} < \Pi_{old,k} \text{ if } \alpha_k > \frac{(2m+3)\left(a+\sum_{\substack{i=1\\ i\neq k}}^{m}\alpha_i\right) + (m+1)\alpha_{m+1}}{2m^2+4m+1}$$

If the new company will have a sufficiently high marginal cost, then the total production will decline, the selling price will increase and firms with sufficiently low marginal costs will increase their profits, while those with higher marginal costs declines in returns.

Case 3

$$\alpha_{m+1} = \frac{a + \sum_{i=1}^{m} \alpha_i}{m+1} \Rightarrow Q_{new} = Q_{old},\ p^*_{new} = p^*_{old},\ \Pi_{new,k} = \Pi_{old,k}\ \ \forall k = \overline{1,m}$$

When entering a new firm with marginal cost exactly determined by the cost of other companies, according to the formula above, both total production and the selling price and profits old companies will remain constant.

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