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An Alternative to the Electre Method

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Abstract: The paper proposes an alternative of the Electre method consisting in replacing the concordance and discordance coefficients with continuous utility functions.

Keywords: Electre; concordance; discordance

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1. Introduction

The ELECTRE method consists in the existence of a number of n alternatives for a decident: V_1, V_2,...,V_n. Let us also consider a number of m criteria C_1, C_2,...,C_m that have each a coefficient of importance (usually subjectively determined) k_1, k_2,...,k_m. For each pair (V_i,C_j) we set a numerical value v_{ij} (if it is a qualitative appreciation of the kind: weak, good, very good etc. we convert it to hierarchy numbers). The problem lies in determining the optimal action variant.

The algorithm consists in first establishing the nature of the method (maximizing or minimizing).

It is important to take into account, at this step, that all the criteria lead to the same nature of the problem. Thus, if the problem is, for example, maximization (minimization), and one or more criteria aim at minimizing (maximizing), the v_{ij} values corresponding to the criterion C_j - in question with the ($-v_{ij}$) values will be replaced.

We then normalize the importance coefficients by the relation: $\nu_i=\frac{k_i}{\sum_{p=1}^{m} k_p}$, $i=\overline{1,m}$ and we therefore have: $\sum_{i=1}^{m} \nu_i=1$.

Then we shall determine the utilities U_{ij} corresponding to the pairs (V_i,C_j) as follows:

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- for the maximization problem: $U_{ij}=\dfrac{v_{ij}-\min\limits_{k=1,\dots,n} v_{kj}}{\max\limits_{k=1,\dots,n} v_{kj}-\min\limits_{k=1,\dots,n} v_{kj}}$;
- for the minimization problem: $U_{ij}=\dfrac{\max\limits_{k=1,\dots,n} v_{kj}-v_{ij}}{\max\limits_{k=1,\dots,n} v_{kj}-\min\limits_{k=1,\dots,n} v_{kj}}$.

and build their table.

Utilities are particularly important from two points of view. On the one hand, it is noted that these are dimensionless (being obtained as ratios between sizes of the same nature) which will allow comparison of different sizes of different natures.

On the other hand, utilities provide an overview of the quantities of each criterion, namely, the closer they are to the problem requirement (maximization or minimization), the utility is closer to 1.

The larger the quantity of the problem, the utility is closer to zero.

It should also be noted that utilities are quantities always in the range: [0,1].

Concordance indicators are calculated as follows:

$$c(V_i,V_j)=\frac{\sum\limits_{\substack{p=1,\dots,m\\ U_{ip}\ge U_{jp}}} k_p}{\sum\limits_{r=1}^{m} k_r}=\sum\limits_{\substack{p=1,\dots,m\\ U_{ip}\ge U_{jp}}} \nu_p$$

Practically, the concordance indicator of variant V_i with V_j is determined by comparing the lines corresponding to V_i and V_j, and where the utility of a variation corresponding to a criterion is greater than or equal to the utility of the other variant for the same criterion, the normalized importance coefficient is added.

We always have: $c(V_i,V_i)=1$, $i=\overline{1,m}$ and $c(V_i,V_j)\in[0,1]$, $i,j=\overline{1,m}$.

We also notice that the concordance indicator $c(V_i,V_j)$ is closer to 1 if the greatest number of utilities of V_i are greater than or equal to the corresponding utilities of V_j (ie, the variant V_i are closer than V_j to the requirements of the problem) and vice versa for the values of concordance close to 0.

The discordance indicators are calculated as follows:

$$d(V_i,V_j)=\max_{p=1,\dots,m}(U_{jp}-U_{ip},0)$$

Practically, the discordance indicator of variant V_i with V_j is determined by comparing the lines corresponding to V_i and V_j, and where the utility of the variant V_j corresponding to a criterion is greater than or equal to the utility of the other V_i variant, for the same criterion, the difference is calculated. The highest value provided by them is determined. Always, we will have:

$d(V_i,V_i)=0$, $i=\overline{1,m}$ and $d(V_i,V_j)\in[0,1]$, $i,j=\overline{1,m}$.

We also notice that the discordance indicator $d(V_i,V_j)$ is closer to 0 if the largest number of utilities of V_i are greater than or equal to the corresponding utilities of V_j (ie, the variant V_i are closer than V_j to the requirements of the problem) and vice versa for the discordance values close to 1.

From the definitions of concordance and discordance indicators we can deduce their general formulas:

$$c(V_i,V_j)=\sum_{p=1}^{m}\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p \text{ , i,j=}\overline{1,m}$$

$$d(V_i,V_j)=\max_{p=1,m}\left(\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})-1)(U_{ip}-U_{jp})\right)\text{, i,j=}\overline{1,m}$$

where the function sgn (signum - lat., sign) is well known:

$$\operatorname{sgn}(x)=\begin{cases}1 \text{ if } x>0;\\0 \text{ if } x=0;\\-1 \text{ if } x<0\end{cases}$$

Indeed, for the concordance indicator we have:

$$c(V_i,V_j)=\sum_{p=1}^{m}\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p=\sum_{\substack{p=1,\dots,m\\U_{ip}>U_{jp}}}\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p+$$

$$\sum_{\substack{p=1,\dots,m\\U_{ip}=U_{jp}}}\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p+\sum_{\substack{p=1,\dots,m\\U_{ip}<U_{jp}}}\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p=\sum_{\substack{p=1,\dots,m\\U_{ip}>U_{jp}}}\operatorname{sgn}(2)\nu_p+$$

$$\sum_{\substack{p=1,\dots,m\\U_{ip}=U_{jp}}}\operatorname{sgn}(1)\nu_p+\sum_{\substack{p=1,\dots,m\\U_{ip}<U_{jp}}}\operatorname{sgn}(0)\nu_p=\sum_{\substack{p=1,\dots,m\\U_{ip}>U_{jp}}}\nu_p+\sum_{\substack{p=1,\dots,m\\U_{ip}=U_{jp}}}\nu_p=\sum_{\substack{p=1,\dots,m\\U_{ip}\geq U_{jp}}}\nu_p\text{ , i,j=}\overline{1,m}$$

and for the discordance:

$$d(V_i,V_j)=\max_{p=1,m}\left(\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})-1)(U_{ip}-U_{jp})\right)=$$

$$\max\left(\max_{\substack{p=1,m\\U_{ip}>U_{jp}}}\left(\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})-1)(U_{ip}-U_{jp})\right),\max_{\substack{p=1,m\\U_{ip}=U_{jp}}}\left(\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})-1)(U_{ip}-U_{jp})\right),\right.$$

$$\left.\max_{\substack{p=1,m\\U_{ip}<U_{jp}}}\left(\operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})-1)(U_{ip}-U_{jp})\right)\right)=$$

$$\max\left(\max_{\substack{p=1,m\\U_{ip}>U_{jp}}}\left(\operatorname{sgn}(0)(U_{ip}-U_{jp})\right),\max_{\substack{p=1,m\\U_{ip}=U_{jp}}}\left(\operatorname{sgn}(-1)(U_{ip}-U_{jp})\right),\max_{\substack{p=1,m\\U_{ip}<U_{jp}}}\left(\operatorname{sgn}(-2)(U_{ip}-U_{jp})\right)\right)=$$

$$\max(\max_{\substack{p=1,m\\U_{ip}<U_{jp}}}\left(U_{jp}-U_{ip}),0\right)=\max_{p=1,\dots,m}(U_{jp}-U_{ip},0)\text{, i,j=}\overline{1,m}\text{.}$$

We shall build a table and we shall pass the concordance indicators to the left, and the discordance to the right of each cell of a table that will have lines and columns of alternatives V_i.

Two p and q values (with complementary probability significance) are set so that $p,q\in(0,1)$ and $p+q=1$ to measure the admitted concordance and discordance limits. So we will say that a variant V_i surpasses a variant V_j if:

$$\begin{cases} c(V_i, V_j) \geq p \\ d(V_i, V_j) \leq q \end{cases}$$

Thus we have: $\begin{cases} c(V_i, V_j) \geq p \\ d(V_i, V_j) \leq q = 1 - p \end{cases}$ $\Rightarrow p \leq \min(c(V_i,V_j), 1-d(V_i,V_j))$.

Computing $\min\limits_{j=\overline{1,n}} c(V_i,V_j)$ and $\min\limits_{j=\overline{1,n}} (1-d(V_i,V_j))$ we obtain:

$p \leq \min(\min\limits_{j=\overline{1,n}} c(V_i,V_j), 1-\max\limits_{j=\overline{1,n}} d(V_i,V_j))$.

The chosen variant is the one for which the maximum of p is obtained.

2. An Alternative to the Indicators

The determintion of the concordance and discordance indicators has the great drawback that it requires comparisons on the components of the utilities of the two alternatives, leading eventually to discontinuous functions.

In the following, we will build a new concordance function that will be not only continuous but also differentiable and also, a function of discordance that will have a continuous character.

Let consider ow, the Heaviside unit's stepping-up function:

$$H(x)=\begin{cases} 0 \ if \ x < 0; \\ \frac{1}{2} \ if \ x = 0; \\ 1 \ if \ x > 0 \end{cases}$$

We have sgn(x)=2H(x)-1.

The function $u(x)=\frac{1}{1+e^{-2kx}}$ approximates differentiable (better and better as k is higher) function H.

The function signum becomes:

$$sgn(x)=2u(x)-1=\begin{cases} -1 \ if \ x < 0; \\ 0 \ if \ x = 0; \\ 1 \ if \ x > 0 \end{cases}$$

Thus: $sgn(x)=\frac{2}{1+e^{-2kx}}-1=\frac{1-e^{-2kx}}{1+e^{-2kx}}=\frac{e^{2kx}-1}{e^{2kx}+1}$

Also $sgn(a-b)=\frac{e^{2k(a-b)}-1}{e^{2k(a-b)}+1}=\frac{e^{2ka}-e^{2kb}}{e^{2ka}+e^{2kb}}$ and $sgn(sgn(a-b)+1)=\frac{e^{2k\frac{e^{2ka}-e^{2kb}}{e^{2ka}+e^{2kb}}}-e^{2k}}{e^{2k\frac{e^{2ka}-e^{2kb}}{e^{2ka}+e^{2kb}}}+e^{2k}}$,

$$sgn(sgn(a-b)-1)=\frac{e^{2k\frac{e^{2ka}-e^{2kb}}{e^{2ka}+e^{2kb}}}-e^{-2k}}{e^{2k\frac{e^{2ka}-e^{2kb}}{e^{2ka}+e^{2kb}}}+e^{-2k}}.$$

From the formula: $c(V_i,V_j)=\sum_{p=1}^{m} sgn(sgn(U_{ip}-U_{jp})+1)\nu_p$, $i,j=\overline{1,m}$ we have therefore:

$$c(V_i,V_j)=\sum_{p=1}^{m} \operatorname{sgn}(\operatorname{sgn}(U_{ip}-U_{jp})+1)\nu_p \sum_{p=1}^{m} \frac{e^{2k\frac{e^{2kU_{ip}}-e^{2kU_{jp}}}{e^{2kU_{ip}}+e^{2kU_{jp}}}}-e^{2k}}{e^{2k\frac{e^{2kU_{ip}}-e^{2kU_{jp}}}{e^{2kU_{ip}}+e^{2kU_{jp}}}}+e^{2k}}\vartheta_p$$

and

$$d(V_i,V_j)=\max_{p=\overline{1,m}}\left(\frac{e^{2k\frac{e^{2kU_{ip}}-e^{2kU_{jp}}}{e^{2kU_{ip}}+e^{2kU_{jp}}}}-e^{-2k}}{e^{2k\frac{e^{2kU_{ip}}-e^{2kU_{jp}}}{e^{2kU_{ip}}+e^{2kU_{jp}}}}+e^{-2k}}(U_{ip}-U_{jp})\right),\ i,j=\overline{1,m}$$

3. Example

Consider the Electre problem:

Table 1

Criterion nature	min	min	min	max
Coefficients of importance	**4**	**2**	**4**	**5**
Alternative/Criterion	**C_1**	**C_2**	**C_3**	**C_4**
V_1	**1805**	**4**	**436**	**38**
V_2	**1458**	**0**	**353**	**15**
V_3	**1177**	**0**	**312**	**36**
V_4	**1109**	**4**	**378**	**21**
V_5	**1669**	**3**	**170**	**13**

Classic solving with Electre method

Table 2

Table recalculation to maximize				
Alternative/Criterion	**C1**	**C2**	**C3**	**C4**
V1	**-1805**	**-4**	**-436**	**38**
V2	**-1458**	**0**	**-353**	**15**
V3	**-1177**	**0**	**-312**	**36**
V4	**-1109**	**-4**	**-378**	**21**
V5	**-1669**	**-3**	**-170**	**13**
min	**-1805**	**-4**	**-436**	**13**
max	**-1109**	**0**	**-170**	**38**
max-min	**696**	**4**	**266**	**25**

Table 3

Utilities				
Normalized coefficients	**0,27**	**0,13**	**0,27**	**0,33**
Alternative/Criterion	**C1**	**C2**	**C3**	**C4**
V1	**0**	**0**	**0**	**1**
V2	**0,5**	**1**	**0,31**	**0,08**
V3	**0,9**	**1**	**0,47**	**0,92**
V4	**1**	**0**	**0,22**	**0,32**
V5	**0,2**	**0,25**	**1**	**0**

Table 4

Table of concordance and discordance indicators

	V_1 (C)	V_1 (D)	V_2 (C)	V_2 (D)	V_3 (C)	V_3 (D)	V_4 (C)		V_4 (D)	V_5 (C)	V_5 (D)
V_1	1	0	0,33	1	0,33	1	0,46		1	0,33	1
V_2	0,67	0,92	1	0	0,13	0,84	0,4		0,5	0,73	0,69
V_3	0,67	0,08	1	0	1	0	0,73		0,1	0,73	0,53
V_4	0,67	0,68	0,6	1	0,27	1	1		0	0,6	0,78
V_5	0,67	1	0,27	0,75	0,27	0,92	0,4		0,8	1	0

Finally:

	min C	1-max D	min
V_1	0,33	0	0
V_2	0,13	0,08	0,08
V_3	0,67	0,47	0,47
V_4	0,27	0	0
V_5	0,27	0	0

The optimal alternative is V_3 (for min=0,47).

The modified Electre method (for k=6)

Table 5

Utilities				
Normalized coefficients	0,27	0,13	0,27	0,33
Alternative/Criterion	C_1	C_2	C_3	C_4
V_1	1	1	1	162754,79
V_2	403,43	162755	41,26	2,61
V_3	49020,8	162755	281,46	62317,65
V_4	162754,79	1	14,01	46,53
V_5	11,02	20,09	162755	1

Table 6

Table with utilities - sgn(U_{ip}-U_{jp})				
Alternative/Criterion	V_1			
	C_1	C_2	C_3	C_4
V_1	0	0	0	0
V_2	1	1	0,95	-1
V_3	1	1	0,99	-0,45
V_4	1	0	0,87	-1
V_5	0,83	0,91	1	-1

Alternative/Criterion	V_2			
	C_1	C_2	C_3	C_4
V_1	-1	-1	-0,95	1
V_2	0	0	0	0
V_3	0,98	0	0,74	1
V_4	1	-1	-0,49	0,89
V_5	-0,95	-1	1	-0,45

Alternative/Criterion	V_3			
	C_1	C_2	C_3	C_4
V_1	-1	-1	-0,99	0,45
V_2	-0,98	0	-0,74	-1
V_3	0	0	0	0
V_4	0,54	-1	-0,91	-1
V_5	-1	-1	1	-1

Alternative/Criterion	V_4			
	C_1	C_2	C_3	C_4
V_1	-1	0	-0,87	1
V_2	-1	1	0,49	-0,89
V_3	-0,54	1	0,91	1
V_4	0	0	0	0
V_5	-1	0,91	1	-0,96

Alternative/Criterion	V_5			
	C_1	C_2	C_3	C_4
V_1	-0,83	-0,91	-1	1
V_2	0,95	1	-1	0,45
V_3	1	1	-1	1
V_4	1	-0,91	-1	0,96
V_5	0	0	0	0

Table 7

Table of concordance and discordance indicators										
	V_1 (C)	V_1 (D)	V_2 (C)	V_2 (D)	V_3 (C)	V_3 (D)	V_4 (C)	V_4 (D)	V_5 (C)	V_5 (D)
V_1	1	0	0,41	1	0,35	1	0,64	1	0,6	1
V_2	0,67	0,92	1	0	0,41	0,84	0,59	0,5	0,73	0,69
V_3	1	0,08	1	0	1	0	1	0,1	0,73	0,53
V_4	0,67	0,68	0,87	1	0,4	1	1	0	0,66	0,78
V_5	0,67	1	0,68	0,75	0,27	0,92	0,48	0,8	1	0

	min C	1-max D	min
V_1	0,35	0	0
V_2	0,41	0,08	0,08
V_3	0,73	0,47	0,47
V_4	0,4	0	0
V_5	0,27	0	0

The optimal alternative is V_3 (for min=0,47).

By testing the accuracy of the algorithm for 100,000 random problems for 5 alternatives and 4 criterions, we obtained the following percentages of overlap between the two methods:

Table 8

k	%
2	96,142
3	96,228
4	96,355
5	96,583
6	96,811
7	96,973
8	97,326
9	97,540
10	97,671
20	98,569

We appreciate (also because exponential increases greatly) from the above table that the recommended value of k is 6.

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