A method of determination of an acquisition program in order to maximize the total utility

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Abstract. This paper solves in a different way the problem of maximization of the total utility. The author uses the diofantic equations (equations in integers numbers) and after a decomposing in different cases, he obtains the maximal utility.

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1 A method of maximization the total utility

Let a consumer which has a budget of acquision of two goods, in value of $S\in\mathbf{N}$ u.m. The prices of the two goods x and y are p_x and $p_y\in\mathbf{N}$ respectively. The marginal utlities corresponding to an arbitrary number of doses are in the following table:

No. of dose	U_{mx}	U_{my}
1	u_{11}	u_{12}
...	...	...
i	u_{i1}	u_{i2}
...	...	...
n	u_{n1}	u_{n2}

We want in what follows to determine the number of doses x, respectivelz b of y such that the total utility: $U_t=\sum_{i=1}^{a}u_{i1}+\sum_{j=1}^{b}u_{j2}$ to be maximal.

Let therefore $S_1\le S$ and the equation:

(1) $ap_x+bp_y=S_1$.

Let denote with $d=(p_x,p_y)$ the greatest common divisor of p_x and p_y. We well know the fact that the equation has entire solutions it is necessary that $d|S_1$. Also, we shall consider: $S_1>S-\min\{p_x,p_y\}$ because if $S_1\le S-\min\{p_x,p_y\}$ with a supplementary unit of x or y, the total utility will grow.

Dividing (1) at d, we have:

(2) $a\frac{p_x}{d}+b\frac{p_y}{d}=\frac{S_1}{d}$

and with the notation: $p'_x=\frac{p_x}{d}$, $p'_y=\frac{p_y}{d}$ follows:

(3) $ap'_x+bp'_y=\frac{S_1}{d}$.

It is well known that for any relative prime numbers $A,B\in\mathbf{N}$ it exist α and $\beta\in\mathbf{Z}$ (determined eventually with the Euclid algorithm) suc that: $\alpha A+\beta B=1$. Like $(p'_x,p'_y)=1$ follows that $\exists\alpha,\beta\in\mathbf{Z}$ such that:

(4) $\alpha p'_x+\beta p'_y=1$, $\alpha p_x+\beta p_y=d$.

We have therefore:

(5) $ap'_x+bp'_y=\frac{S_1}{d}(\alpha p'_x+\beta p'_y)$

or, in other words:

(6) $p'_x(a-\frac{S_1}{d}\alpha)=p'_y(\frac{S_1}{d}\beta-b)$.

Because $(p'_x,p'_y)=1$ follows from (6) that it exist $k\in\mathbf{Z}$ such that:

(7) $a-\frac{S_1}{d}\alpha=kp'_y$; $\frac{S_1}{d}\beta-b=k\,p'_x$

or:

(8) $a=kp'_y+\frac{S_1}{d}\alpha$; $b=-k\,p'_x+\frac{S_1}{d}\beta$.

We can easily write (8) like:

(9) $a=\frac{kp_y+S_1\alpha}{d}$; $b=\frac{-kp_x+S_1\beta}{d}$.

We have, a,b≥0, and from (1): $a\le\frac{S_1}{p_x}$, $b\le\frac{S_1}{p_y}$.

From (9) we have:

$$(10)\qquad \begin{cases} k\ge-\frac{S_1\alpha}{p_y}\\ k\le\frac{S_1\beta}{p_x}\\ k\le\frac{S_1(d-\alpha p_x)}{p_xp_y}\\ k\ge\frac{S_1(\beta p_y-d)}{p_xp_y}\end{cases}$$

From (4) and (10) follows:

$$(11)\qquad \begin{cases} k\ge-\frac{S_1\alpha}{p_y}\\ k\le\frac{S_1\beta}{p_x}\\ k\le\frac{S_1\beta}{p_x}\\ k\ge-\frac{S_1\alpha}{p_y}\end{cases}$$

therefore:

$$(12)\qquad k\in\left[-\frac{S_1\alpha}{p_y},\frac{S_1\beta}{p_x}\right]\cap\mathbf{N}=S_1\left[-\frac{\alpha}{p_y},\frac{\beta}{p_x}\right]\cap\mathbf{N}.$$

The length of the interval: $\left[-\frac{S_1\alpha}{p_y},\frac{S_1\beta}{p_x}\right]$ is $\frac{S_1\beta}{p_x}+\frac{S_1\alpha}{p_y}=\frac{S_1(\beta p_y+\alpha p_x)}{p_xp_y}=\frac{S_1d}{p_xp_y}$. Because $p_xp_y=d[p_x,p_y]$ (the least common multiple of the two numbers) we obtain that the length of the interval is: $\frac{S_1}{[p_x,p_y]}$. Will be exist therefore $\left[\frac{S_1}{[p_x,p_y]}\right]$+1 entire values of k (where with [z] we have noted the entire part of z) who verify the acceptability conditions.

Let therefore: $a_k=\frac{kp_y+S_1\alpha}{d}$, $b_k=\frac{-kp_x+S_1\beta}{d}$ with k upper determined.

We have: $a_{k+1}=\frac{(k+1)p_y+S_1\alpha}{d}=a_k+\frac{p_y}{d}$ and $b_{k+1}=\frac{-(k+1)p_x+S_1\beta}{d}=b_k-\frac{p_x}{d}$.

Because: $U_{t,k}=\sum_{i=1}^{a_k} u_{i1} + \sum_{j=1}^{b_k} u_{j2}$ şi $U_{t,k+1}=\sum_{i=1}^{a_{k+1}} u_{i1} + \sum_{j=1}^{b_{k+1}} u_{j2} = \sum_{i=1}^{a_k} u_{i1} + \sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} + \sum_{j=1}^{b_k} u_{j2} - \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2} =$

$U_{t,k}+\sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} - \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2}$ where $U_{t,k}$ is the total utility corresponding to k.

If exist k such that: $U_{t,k+1}<U_{t,k}$ then:

$$\sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} - \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2} <0 \text{ or other:}$$

$$\sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} < \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2} .$$

We have: $\sum_{i=a_{k+1}+1}^{a_{k+1}+\frac{p_y}{d}} u_{i1} < \sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1}$ because the both terms of sum have $\frac{p_y}{d}$ components, and the marginal utilities are a descending range, a_k being ascending and analogously: $\sum_{j=b_{k+1}-\frac{p_x}{d}+1}^{b_{k+1}} u_{j2} > \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2}$ because the both terms of sum have $\frac{p_x}{d}$ components, and the marginal utilities are a descending range, b_k being descending,

We have now:

$$U_{t,k+2}= U_{t,k+1}+\sum_{i=a_{k+1}+1}^{a_{k+1}+\frac{p_y}{d}} u_{i1} - \sum_{j=b_{k+1}-\frac{p_x}{d}+1}^{b_{k+1}} u_{j2} < U_{t,k+1}+\sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} - \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2} = U_{t,k}+2\left(\sum_{i=a_k+1}^{a_k+\frac{p_y}{d}} u_{i1} - \sum_{j=b_k-\frac{p_x}{d}+1}^{b_k} u_{j2}\right)< U_{t,k}.$$

Like a conclusion, the range of total utilities, once it reach a local maximum for a k, it reach in that point a global maximum.

2 Example

No. of dose	U_{mx}	U_{my}
1	10	20
2	8	16
3	7	15
4	6	14
5	5	13
6	4	10
7	3	8
8	2	7

$p_x=4$, $p_y=6$, S=33.

Solution

We have $\min\{p_x,p_y\}=4$, therefore $S_1\in(29,33]$. Like $(p_x,p_y)=2$ follows that $S_1\in\{30,32\}$.

We have now: $4\cdot(-1)+6\cdot 1=2$ therefore α=-1 and β=1. From (12),we obtain: $k\in S_1\left[\frac{1}{6},\frac{1}{4}\right]\cap\mathbf{N}$.

Like a conclusion:

- $S_1=30\Rightarrow k\in\left[\frac{30}{6},\frac{30}{4}\right]\cap\mathbf{N}=\{5,6,7\}$;

- $S_1=32 \Rightarrow k \in \left[\frac{32}{6}, \frac{32}{4}\right] \cap \mathbf{N} = \{6,7,8\}$.

  From the upper relations:

  - $S_1=30 \Rightarrow a_k=\frac{6k-30}{2}=3k-15$, $b_k=\frac{-4k+30}{2}=-2k+15$, $k \in \{5,6,7\}$;
  - $S_1=32 \Rightarrow a_k=\frac{6k-32}{2}=3k-16$, $b_k=$ └ $=-2k+16$, $k \in \{6,7,8\}$.

  It follows:

  - $S_1=30 \Rightarrow$
    - $k=5 \Rightarrow a_5=0$, $b_5=5 \Rightarrow U_{t,5}=20+16+15+14+13=78$
    - $k=6 \Rightarrow a_6=3$, $b_6=3 \Rightarrow U_{t,6}=10+8+7+20+16+15=76$
    - $k=7 \Rightarrow a_7=6$, $b_7=1 \Rightarrow U_{t,7}=$non computing!
  - $S_1=32 \Rightarrow$
    - $k=6 \Rightarrow a_6=2$, $b_6=4 \Rightarrow U_{t,6}=10+8+20+16+15+14=83$
    - $k=7 \Rightarrow a_7=5$, $b_7=2 \Rightarrow U_{t,7}=10+8+7+6+5+20+16=72$
    - $k=8 \Rightarrow a_8=8$, $b_8=0 \Rightarrow U_{t,8}=$ non computing!

  Finally, the maximal utility will be $U_t=83$ for 2 goods x and 4 goods y.